Topological terms in Feynman path integrals

Handout to the presentation in the framework of the proseminar on condensed matter theory

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May 9, 2009

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1 Introduction

When Feynman path integrals occur, we are most of the time not able to evaluate them directly. Nevertheless it is interesting to study the properties of the integrals as some terms may have special attributes that we can use. Actually we are going to find that some terms in certain setups can be pulled out of the integral, what we use to evaluate the Feynman integrals. First, we are going to look at the particle on a ring problem, which is very easy to solve and is a good subject for our study.

Figure 1: Particle on a ring

2 Particle on a ring

Consider a particle in one dimension with periodic boundary conditions, formed to a ring. To make it more interesting, let us assume that the ring is threaded by a magnetic flux Φ. We choose a angular coordinate $\phi \in [0, 2\pi]$ and set $\hbar = c = e = 1$, and the radius and mass of the particle to unity ($m = r = 1$). Then, the Hamiltonian has the following form:

$$\hat{H} = \frac{1}{2}(-i\partial_\phi - A)^2$$

with $A := \Phi/\Phi_0$ where $\Phi_0 = hc/e = 2\pi$ denotes the magnetic flux quantum.

2.1 The eigensystem

First we solve this problem directly, without use of the Feynman path integral. So we are interested in the eigenfunctions Ψ and eigenvalues E of $\hat{H}$:

$$\begin{cases} \frac{1}{2}(-i\partial_\phi - A)^2\Psi(\phi) = \hat{H}\Psi(\phi) = \epsilon\Psi(\phi) \\ \Psi(0) = \Psi(2\pi) \end{cases}$$

Considering the derivative $\frac{d}{d\phi}\exp(in\phi) = in\exp(in\phi)$, its easy to see that we have the following solutions:

$$\Psi_n(\phi) = \frac{1}{\sqrt{2\pi}}\exp(in\phi), \qquad \epsilon_n = \frac{1}{2}(n - A)^2, \qquad n \in \mathbb{Z}$$

2.2 Solution of the classical problem

To solve an analogical problem in classical mechanics, we first derive the Hamilton function $H(q, p)$ of the Hamilton operator $H(\hat{q}, \hat{p})$ which is a simple replacement: $\hat{q} \mapsto q \equiv \phi$, $-i\partial_\phi = \hat{p} \mapsto p = \dot{\phi} - iA$. The corresponding Lagrange function to his Hamiltonian is:

$$L(\phi, \dot{\phi}) = \frac{1}{2}\dot{\phi}^2 - iA\dot{\phi}$$

which can be easily checked by building $H = L(\theta, p) - \theta p$ where $p = \partial L/\partial\theta = \dot{\theta} - iA$

2.3 Formulating the Feynman path integral

With the classical solution and the eigensystem of $\hat{H}$ we can now formulate the Feynman path integral of the partition function and calculate it explicitly. It is to be noted, that this is one rare case — normally it wont be possible to calculate the partition function per se.

From a previous talk we know how to forumlate the partiton function as a Feyman path integral:

$$\mathcal{Z} = \int_0^{2\pi} d\theta < \theta | e^{-\beta \hat{H}} | \theta > = \int_0^{2\pi} d\theta \int_{\theta(\beta)=\theta(0)=\theta} D\theta(\tau) \exp\left[- \int_0^{\beta} d\tau \frac{I}{2} \dot{\theta}^2 \right]$$

As we are calculating the trace, we must, because of the $< \theta | e^{-\beta \hat{H}} | \theta >$-term, integrate over all paths starting and ending at the same angle θ. This is very important to understand, as we will get our topological terms because of this fact.

$$\mathcal{Z} = \sum_{n=-\infty}^{\infty} \exp\left[-\beta \frac{n^2}{2I} \right]$$

3 Topology

3.1 Introducing example

In topology, we study sets of points and how two sets are similar to each other. It's not only important how many elements there are in a set, but also what neighbours each element has. To clarify we discuss an easy example of two sets that are not similar to each other in the sense of homotopy:

Let there be the two sets:

$$\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \dots\}$$
$$\mathbb{Q} := \{0, \pm 1, \pm \frac{1}{2}, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{3}{2}, \pm 3, \dots\}$$

From linear algebra we know, that there is a bijective map between those two sets, so they have the same order of number of elements. But we also know from calculus, that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, where $\mathbb{Z}$ is not. So for each two elements of $\mathbb{Q}$, there are an infinite number of elements "between" those two. $\mathbb{Z}$ doesn't have this attribute. If we want to preserve the neighbour-attributes of points when transitioning from one set to the other, we cannot find such a transition for these two sets — we say the two sets $\mathbb{Q}$ and $\mathbb{Z}$ are not homotopic to each other.

The same holds for the two sets of a sphere S^2 and a torus $\mathbb{T}^2$; they are not homotopic to each other; whereas the torus is homotopic to a coffee mug.

Now let us define hpmotopy in a more mathematical way.

3.2 Homotopy

Consider a field $\phi : S^d \to T$ where $T(= G/H)$ is the target space (cf. section 2 where we considered $\phi : S^1 \to S^1 \simeq O(2)$). We say two fields ϕ_1,ϕ_2 of this form are topologically equivalent, or **homotopic**, if they can continuously deformed into each other, so if there exists a mapping of the form:

$$\phi : S^d \times [0,1] \to T \quad \text{continuous}$$
$$(z,t) \mapsto \hat{\phi}(z,t)$$

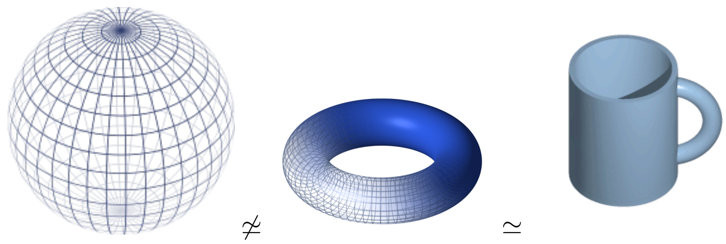

Figure 2: A sphere, a torus and a coffee mug ([1], [2], [3])

where $\hat{\phi}(\cdot, 0) \equiv \phi_1$ and $\hat{\phi}(\cdot, 1) \equiv \phi_2$. Such a function is called a homotopy.

We can now collect all fields that are homotopic to a field ϕ and put them into a set and call it $[\phi]$. This set is called the **equivalence class** of ϕ. All equivalence classes $\{[\phi]\}$ of mappings $\phi : S^d \to T$ build a group, called the d**th homotopy group**, $\pi_d(T)$.

In the case of the particle on a ring, we had $\phi : S^1 \to S^1$. Each field belongs to a equivalence class $[\phi]$ that can be characterised by its winding number, as each field can be continuously deformed to any other field that has the same winding number. We have now the equivalence classes $\{[\phi_0], [\phi_1], [\phi_{-1}], [\phi_2], \dots\}$ which is isomorphic (as a group) to $\mathbb{Z}$. So, we have $\pi_1(S^1) \simeq \mathbb{Z}$. That's why we were able to sum over the different paths in the action of the particle on a ring! It may be also interesting, that $\pi_1(S^2) \simeq \emptyset$, so each closed curve on a sphere can be continuously deformed to a point.

In contrast to particle physics, where space and time are intertwined by relativistic covariance, in condensed matter physics time is typically compactified to a circle separately, so our base manifold is $\mathcal{M} \simeq S^1 \times S^d$ rather than $\mathcal{M} \simeq S^{d+1}$ which behaves totally differently.

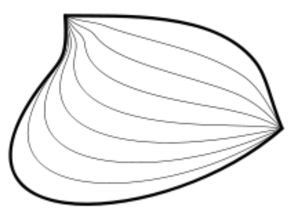

Figure 3: A continuous deformation of two fields[5]

4 Differential geometry

4.1 Differentiable manifolds

In the previous section we talked about manifolds but have not defined nor explained what we really mean by that. We are going to catch up on that and introduce manifolds in a general way.

In words, a **manifold** is a set of points, that locally behaves like the Euclidean space, although the manifold as a whole may not. On a differentiable manifold, we can even do calculus (Infinitesimalrechnung) and a sympletic manifold can be used to model the phase space of a Hamiltonian system. To represent a manifold in term of coordinates we need a collection of charts, an atlas, that locally map our points to the Euclidean space. We can then, by using the inverse mapping function, go from a point on our manifold to the Euclidean space, do calculus, and then map the result back to our manifold.

Definition 1 (Manifolds) *The three ingredients:*
A **chart** (U_k, ϕ_k) *is a homeomorphism* ϕ_k *from an open subset* $U_k \subset \mathcal{M}$ *to an open subset of* $\mathbb{R}^n$.
An **atlas** $\mathcal{A}$ *is a collection of charts* $\{(U_k, \phi_k)\}$ *that cover the whole set* $\mathcal{M}$.
A **manifold** *is a second countable Hausdorff space with an atlas (countable base).*

So for each point $p \in \mathcal{M}$ there is an open set U that is homeomorphic to a open subset of $\mathbb{R}^n$. That gives us a local system of coordinates, as we find coordinates of $\mathbb{R}^n$ for each point.

A point $p \in \mathcal{M}$ can have different charts applicable to it - those charts do not need to give the same result; in general: $\phi_k(p) \neq \phi_l(p)$, so a point can have different coordinates representations. To go from one coordinate representation (of chart ϕ_k) to another (of chart ϕ_l), we can use $(\phi_l \circ \phi_k^{-1})(P)$, where $P \in \mathbb{R}^n$ are the coordinates of the point p under the chart ϕ_k.

The map $\phi_{lk} := \phi_l \circ \phi_k^{-1}|_{\phi_k(U_l \cap U_k)}$ is called transition map and must be a diffeomorphism (invertible and differentiable). If all the transition maps fulfill that requirement, we call the manifold a **differentiable manifold**. On such a manifold we can define the tangent space of a point.

4.2 Tangent space $T_p\mathcal{M}$

Let $\mathcal{M}$ be a differentiable manifold. Because we have an atlas, we can define paths on our manifold by using the inverse coordinate map. For example, let $X = \phi_k(U_k) \subset \mathbb{R}^n$, then every continuous function $\nu : I \subset \mathbb{R} \to X$ induces a path on $U_k \subset \mathcal{M}$ by $\phi_k^{-1} \circ \nu : \mathbb{R} \to U_k$. With the same argument we can define functions $f : U_k \to \mathbb{R}$ by combining real functions with the coordinate maps. We call $\mathcal{C}(U_k)$ the space of all smooth, real-valued function on U_k.

Let us now consider such a function $f : U \to \mathbb{R}$ defined on an open neighborhood U_p of $p \in \mathcal{M}$, and a path $\gamma : \mathbb{R} \to \mathcal{M}$ with $\gamma(0) = p$. With the help of these two functions, we are able to define tangent vectors:

Definition 2 *Let* $p \in \mathcal{M}$, γ *and* f *be as above. A* **tangent vector** $\mathbf{v}_p^\gamma(f)$ *is defined as:*

$$\mathbf{v}_p^\gamma(f) \equiv d_s|_{s=0} f(\gamma(s))$$

Keep in mind, that $f \circ \gamma : \mathbb{R} \to \mathbb{R}$, so the derivation d_s is properly defined. Also the tangent "vector" $\mathbf{v}_p^\gamma$ is really a functional, that maps functions f to $\mathbb{R}$. That may be a bit strange but its utility will become apparent, as these functionals can be interpreted as a vector in the sense of algebra. Also notice, that the assignment "curve $\mapsto$ tangent vector" is not unique, as two different paths γ_1 and γ_2 *that have the same tangent at* p ($\mathbf{v}_p^{\gamma_1}(f) = \mathbf{v}_p^{\gamma_2}(f)$) lead to the same tangent vectors $\mathbf{v}_p^{\gamma_1} = \mathbf{v}_p^{\gamma_2}$.

The set of all possible tangent vectors at a point p is called the **tangent space** $T_p\mathcal{M}$ of p. It is not very intuitive, however given a coordinate function ϕ we can meet the standard identification "vector $\leftrightarrow$ n-component object" familiar from linear algebra (omitting the subscript p, remember $\gamma(0) = p$):

$$\begin{aligned}
\mathbf{v}^\gamma(f) &= d_s|_{s=0}(f \circ \gamma)(s) \\
&= d_s|_{s=0} \underbrace{(f \circ \phi^{-1})}_{\mathbb{R}^n \to \mathbb{R}} \circ \underbrace{(\phi \circ \gamma)}_{\mathbb{R} \to \mathbb{R}^n}(s) \\
&= \sum_{i=1}^{n} \partial_i (f \circ \phi^{-1}) \left[d_s|_{s=0}(\phi \circ \gamma)(s) \right]_i \\
&\equiv \sum_{i=1}^{n} \partial_i f \; v_i^\gamma
\end{aligned}$$

where $\partial_i f := \partial_i(f \circ \phi^{-1})$ is the ordinary partial derivative, and $v_i^\gamma := d_s|_{s=0}\phi^i(\gamma(s))$ is the ith component of $\mathbf{v}^\gamma$ in the coordinate representation implied by ϕ. Each smooth, real-valued function f induces a natural base of $T_p\mathcal{M}$: $\mathbf{e}_i(f) := \partial_i f$ as we have seen above. Now we can identify our "algebraic" tangent vectors as a vector $\vec{v}^\gamma \in \mathbb{R}^n$ with components v_i^γ, independent of f!

The union $T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}$ of all tangent spaces is called **tangent bundle** of the manifold $\mathcal{M}$, which is a differentiable tangent space itself! Its elements are given by $(p, \mathbf{v}_p) \in T\mathcal{M}$ where $p \in \mathcal{M}$ and $\mathbf{v}_p \in T_p\mathcal{M}$.

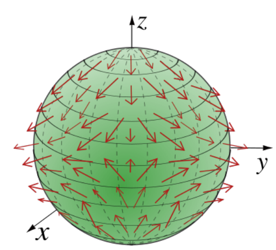

Figure 4: Vector field on S^2, [4]

Finally, we can define **vector fields**. Those are mappings:

$$\begin{aligned}
\mathbf{v} : \mathcal{M} &\to T\mathcal{M} \\
p &\mapsto (p, \mathbf{v}_p)
\end{aligned}$$

4.3 Differential forms

Now that we have defined tangent spaces, we can finally define forms, especially differential forms. These are the mathematical constructs to replace the small path elements "dx" that physicists love to use. With their help one can also generalize the integration $\int dx$ to expressions like $\int \omega$ where ω is a form. Even though we are not going to do that, this can be the base of a deeper understanding of calculus on generalised spaces, i.e. differentiable manifolds.

A 1**-form** ω_p is a *linear* mapping $\omega_p : T_p\mathcal{M} \to \mathbb{R}$ — so a 1-form is a function that maps each tangential vector to a real number. If we now smoothly extend ω_p to be defined on the whole manifold $\mathcal{M}$ we get a **differential** 1**-form** (or just 1-form). We denote the set of all differential 1-forms of a manifold $\mathcal{M}$ by $\Lambda^1(\mathcal{M})$.

As each 1-form must be linear, we can find a base for these mappings. Given an atlas with a system of local coordinate mappings ϕ we can express a 1-form ω as a linear combination of the base $(d\phi^1, \ldots, d\phi^n)$:

$$\omega(\mathbf{v}) = \sum a_i(\mathbf{v}_i)\, d\phi^i(\mathbf{v})$$

where $a_i := \omega(\mathbf{e}_i)$.

5 Θ-terms

6 Quantum spin chain

References

[1] Geek3, (GNU FDL), en.wikipedia.org/wiki/File:Sphere_wireframe.svg

[2] Kieff, (PD), en.wikipedia.org/wiki/File:Torus.png

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